EXPLORING THE VOLUME OF DOUGHNUT/DONUT

Introduction

A donut as a dessert is my first choice. I can have it any time. The shape of it has radial symmetry. When I learned symmetrical shapes (surfaces) like cylinder, sphere, cone etc in mathematics classes, these shapes were associated with volume and surface area. So, an obvious question struck my mind, what is the volume of donut and how it can be derived. The exploration took me to integral calculus and the integral calculus further introduced me the topic of volume of revolution. The topic suggested that the volume of revolution can obtained by revolving curves around an axis and but the shapes generated are symmetrical. Although Donut is symmetrical but the big question is, which is that curve when revolved around an axis produces a Donut. A little effort and searching resolved the query and the curve was a circle. Therefore I could aim my endeavour in finding the volume of Donut. The next question that arises is that donut shapes are not exact circle but the best circle can be approximated through Least square approximation.

Objective

In this exploration I intend to find the volume of Donut. Once the volume formula is known I would exploit least approximation method to find the best circle that imitates the real Donut available in market. Donut is symmetrical in shape and resembles Torus, as is seen in the Figure 1 below.

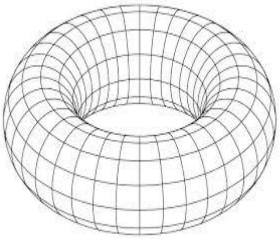

Figure 1

So the volume of a Donut can be looked into as if one is finding the volume of a Torus. Mathematically, a Torus is obtained by revolving a circle around an axis. The idea is made clear through the Figure 2.

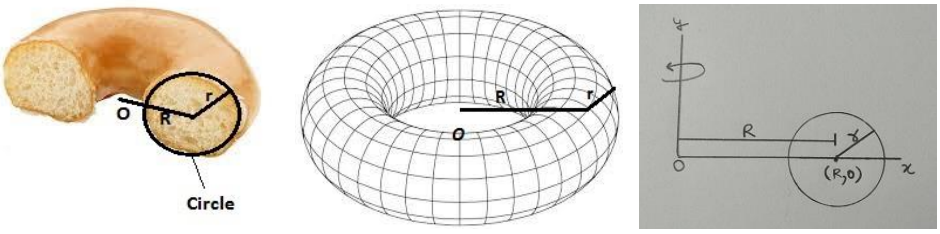

Figure 2

Method

When the circle $(x - R)^2 + y^2 = r^2$ is revolved around y - axis, the Torus is produced as is seen the Figure 3.

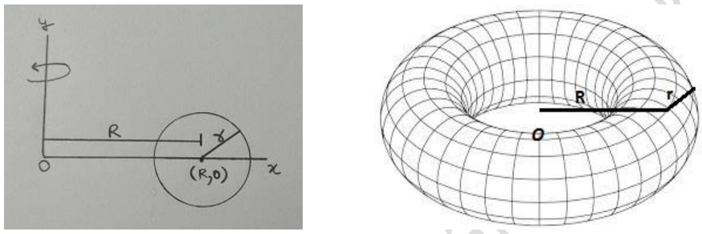

Figure 3

To find the volume of the Torus, literature serves two methods, Washer method and Shell method. The two methods are general in application but in present context they are worked for Torus.

1. **Washer Method**

Imagine a horizontal strip (Figure 4) that revolves around y-axis.

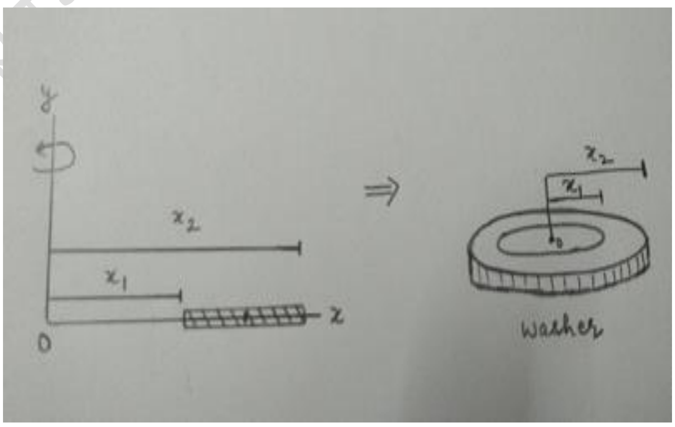

Figure 4

The shape obtained is of washer. Washer is like a disc with a hole inside. The volume of washer can be understood with the help of cylinder of height h and is given as $\pi(x_2)^2h - \pi(x_1)^2h$. This idea is the basis of washer method in finding volume of Torus (Figure 5).

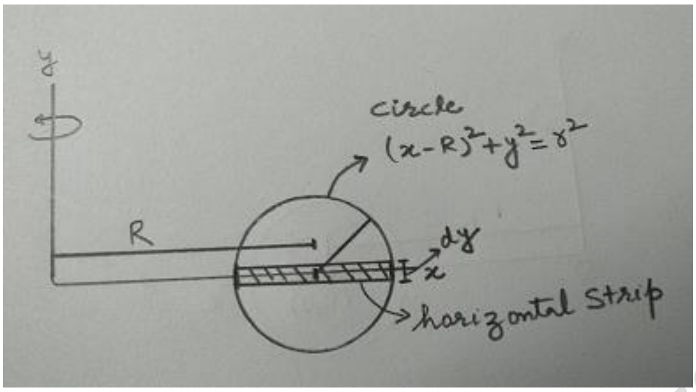

Figure 5

The washers of changing radius pile up to give Torus as is observed in Figure 6.

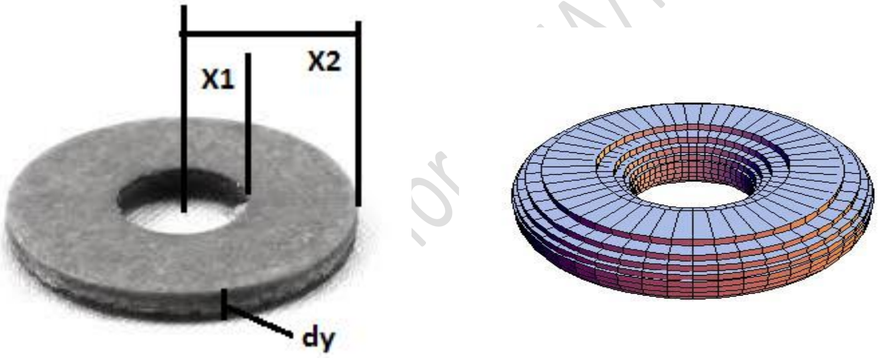

Figure 6

Therefore the volume is given as

$$V = \pi \int_{-r}^{r} [(x_2)^2 - (x_1)^2]dy$$

Where

$(x-R)^2 + y^2 = r^2$

$=> (x-R)^2 = r^2 - y^2$

$=> \quad x - R = \pm\sqrt{r^2 - y^2}$

$=> \quad x = R \pm \sqrt{r^2 - y^2}.$

So we have $=> \quad x_2 = R + \sqrt{r^2 - y^2} \; and \; x_1 = R - \sqrt{r^2 - y^2}$.

The volume the is given as

$$V = \pi \int_{-r}^{r} \left[\left(R + \sqrt{r^2 - y^2}\ \right)^2 - \left(R - \sqrt{r^2 - y^2}\ \right)^2 \right] dy$$

$$=> 4\pi R \int_{-r}^{r} \sqrt{r^2 - y^2}\ dy$$

$$=> 8\pi R \int_{0}^{r} \sqrt{r^2 - y^2}\ dy$$ (since $\sqrt{r^2 - y^2}$ is an even function)

$$=> 8\pi R \left[\frac{y}{2}\sqrt{r^2 - y^2} + \frac{r^2}{2} sin^{-1}\frac{y}{x}\right]_0^r = 2\pi^2 r^2 R\ .$$

2. **Shell Method**

Now imagine a vertical strip, as is shown in Figure7 that revolves around y-axis.

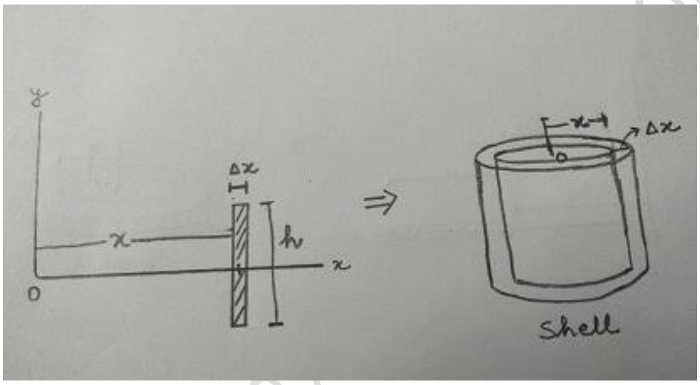

Figure 7

The shape obtained is of a shell. The volume of this shell can again be understood in the form of concentric cylinders of height h. The volume of the shell is

$$=> \quad \pi(x + \Delta x)^2 h - \pi(x)^2 h$$

$$=> \quad \pi h[x^2 + (\Delta x)^2 + 2x\ \Delta x\Delta -\ x^2] = \pi h[(\Delta x)^2 + 2x\ \Delta x].$$

Now if ⊗x is small then $(\Delta x)^2 \rightarrow 0$, Therefore the volume takes the form $2\pi hx\Delta x$. The idea explained forms the basis of Shell method. The volume explained above becomes the element to be integrated for finding the volume of Torus. The volume is understood with help of Figure 8.

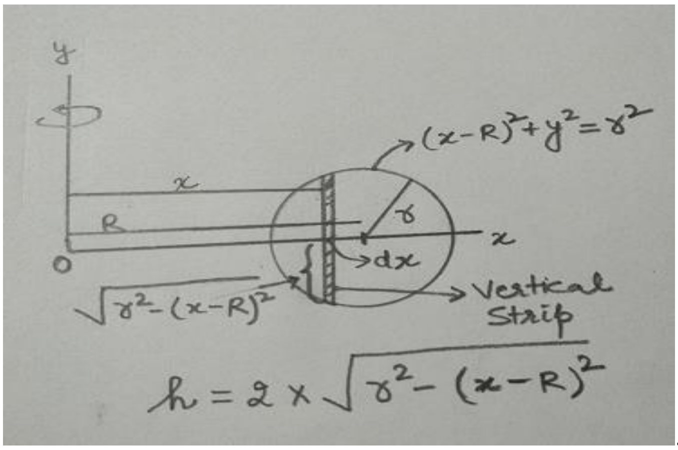

Figure 8

The shells of this kind give the volume of torus as is seen in Figure 9.

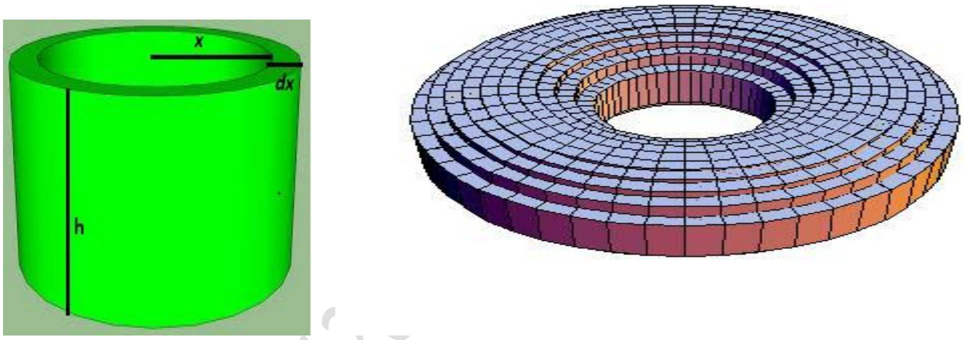

Figure 9

The volume is given as

$$V = 2\pi \int_{R-r}^{R+r} x \times 2\sqrt{r^2 - (x-R)^2}\, dx = 4\pi \int_{R-r}^{R+r} x\sqrt{r^2 - (x-R)^2}\, dx$$

To integrate let $x - R = v \quad then \quad dx = dv$. The volume is

$$V = 4\pi \int_{-r}^{r} (v+R)\sqrt{r^2 - v^2}\, dv$$

$$V = 4\pi \int_{-r}^{r} v\sqrt{r^2 - v^2}\, dv + 4\pi \int_{-r}^{r} R\sqrt{r^2 - v^2}\, dv$$

The first integration above is zero, as it is an odd function. So,

$$V = 4\pi \int_{-r}^{r} R\sqrt{r^2 - v^2}\, dv = 8\pi \int_{0}^{r} R\sqrt{r^2 - v^2}\, dv = 8\pi R \times \frac{\pi r^2}{4} = 2\pi^2 r^2 R.$$

The volume of Donut which is approximated as Torus has volume $= 2\pi^2 r^2 R$.

Least Square Approximation of best Circle

Equation of circle is of the form as is seen in Figure 10 is

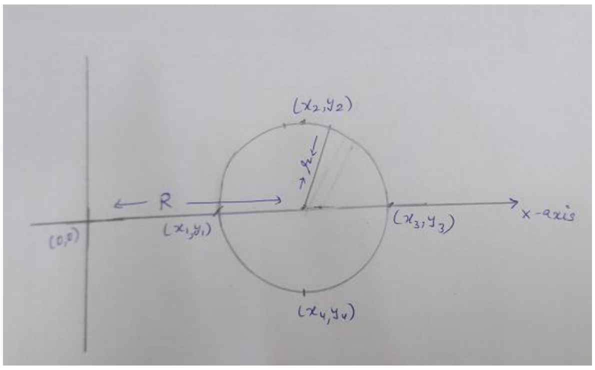

Figure 10

$$(x-R)^2+y^2=r^2$$

$$or \qquad y^2=(r^2-R^2)+2Rx-x^2$$

So we take the equation of circle of form (for simplicity)

$$y^2=a+bx-x^2$$

Where a and b are parameters

For the best circle passing through the points$(x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)$, I take help of least square approximation method.

For point (x_1,y_1), we have

$$y^2=a+bx_1-{x_1}^2$$

But value in fact the value is ${y_1}^2$. So squared error say is $E_1=((a+bx_1-{x_1}^2)-{y_1}^2)^2$.

Similarly for other points

$$E_2=((a+bx_2-{x_2}^2)-{y_2}^2)^2$$

$$E_3=\left((a+bx_3-{x_3}^2)-{y_3}^2\right)^2$$

$$E_4=\left((a+bx_4-{x_4}^2)-{y_4}^2\right)^2$$

Total Error $E = E_1 + E_2 + E_3 + E_4$.

The total error has to be minimized to get the best circle possible.

So $E = \left((a + bx_1 - x_1{}^2) - y_1{}^2\right)^2 + \left((a + bx_2 - x_2{}^2) - y_2{}^2\right)^2 + \left((a + bx_3 - x_3{}^2) - y_3{}^2\right)^2 + \left((a + bx_4 - x_4{}^2) - y_4{}^2\right)^2$

Now problem is to minimize *E* with respect to parameter *a* and *b* one can use necessary conditions of optimization i.e.

$$\frac{\partial E}{\partial a} = 2[(a + bx_1 - x_1{}^2) - y_1{}^2] + 2[(a + bx_2 - x_2{}^2) - y_2{}^2] + 2[(a + bx_3 - x_3{}^2) - y_3{}^2]$$

$$\frac{\partial E}{\partial b} = 2[(a + bx_1 - x_1{}^2) - y_1{}^2]x_1 + 2[(a + bx_2 - x_2{}^2) - y_2{}^2]x_2 + 2[(a + bx_3 - x_3{}^2) - y_3{}^2]x_3 + 2[(a + bx_4 - x_4{}^2) - y_4{}^2]x_4$$

Now $\frac{\partial E}{\partial a} = 0$ and $\frac{\partial E}{\partial b} = 0$ are the necessary condition of extreme point , then

$$4a + b(x_1 + x_2 + x_3 + x_4) = (x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2) + (y_1{}^2 + y_2{}^2 + y_3{}^2 + y_4{}^2)$$

and

$$a(x_1 + x_2 + x_3 + x_4) + b(x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2) = (x_1{}^3 + x_2{}^3 + x_3{}^3 + x_4{}^3) + (x_1 y_1{}^2 + x_2 y_2{}^2 + x_3 y_3{}^2 + x_4 y_4{}^2)$$

Now we have two linear equations in *a* and *b* so the values of *a* and *b* can be uniquely determined.

$$\begin{bmatrix} 4 \sum x_i \sum x_i \sum x_i{}^2 \end{bmatrix} [a\ b\,] = \begin{bmatrix} \sum x_i{}^2 + \sum y_i{}^2 \sum x_i{}^3 + \sum x_i y_i{}^2 \end{bmatrix}$$

I have taken a Donut and took apart a slice. The slice is place on graph paper (Figure 11) and four points on its circumference are taken (Figure 12),

Figure 11

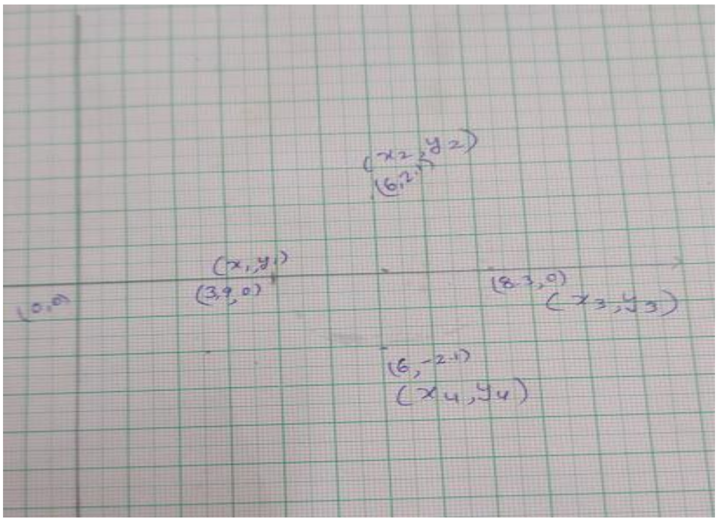

Figure 12

x(cm)	3.9	6	8.3	6
y(cm)	0	2.1	0	-2.1

Therefore using the above derived formula, I have

$$[4\ 24.2\ 24.2\ 156.1\ ][a\ b\ ] = [164.92\ 1116.026\ ]$$

So that we find the value of a and b are

$a = -32.6025,\ b = 12.204.$

Now the equation $y^2 = a + bx - x^2$ takes the form $y^2 = -32.6025 + 12.204x - x^2$,

and is transformed into the form

$$y^2 = (r^2 - R^2) + 2Rx - x^2.$$

I get the values of r and R which are $r = 2.15cm,\ R = 6.102cm$. The values of r and R can be substituted in the derived formula of Volume for Torus. The Volume then is

$$V = 2 \times \pi^2 \times (2.15)^2 \times 6.102 = 556.453cm^3.$$

Conclusion

The volume of the Donut has been derived using Washer and shell method considering it has a shape of Torus. The circle that best fits the real donut has been done with the help of least square approximation method. This exploration has given me an opportunity to study new methods for finding the volume and Least square approximation. Integral calculus uses single definite integral to find the volume of symmetrical shapes. However if the shapes or surfaces are asymmetrical, integral calculus uses multiple integral. The idea of Washer and Shell method can be extended to different symmetrical surfaces.

References

http://www.math.tamu.edu/~tkiffe/calc3/revolution3/revolution3.html

Thomas, G.B.Jr and Weir, M.D., “Thomas Calculus”, Pearson.

Kreyzig, E. “Advanced Engineering Mathematics”, Wiley.