International Baccalaureate
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Diploma Programme

Mathematics HL and further mathematics HL formula booklet

For use during the course and in the examinations
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Contents

Prior learning

Area of a parallelogram	$A = b \times h$, where b is the base, h is the height
Area of a triangle	$A = \frac{1}{2}(b \times h)$, where b is the base, h is the height
Area of a trapezium	$A = \frac{1}{2}(a + b)h$, where a and b are the parallel sides, h is the height
Area of a circle	$A = \pi r^2$, where r is the radius
Circumference of a circle	$C = 2\pi r$, where r is the radius
Volume of a pyramid	$V = \frac{1}{3}$(area of base $\times$ vertical height)
Volume of a cuboid	$V = l \times w \times h$, where l is the length, w is the width, h is the height
Volume of a cylinder	$V = \pi r^2 h$, where r is the radius, h is the height
Area of the curved surface of a cylinder	$A = 2\pi rh$, where r is the radius, h is the height
Volume of a sphere	$V = \frac{4}{3}\pi r^3$, where r is the radius
Volume of a cone	$V = \frac{1}{3}\pi r^2 h$, where r is the radius, h is the height
Distance between two points (x_1, y_1) and (x_2, y_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
Coordinates of the midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
Solutions of a quadratic equation	The solutions of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Core

Topic 1: Algebra

1.1	The *n*th term of an arithmetic sequence	$u_n = u_1 + (n-1)d$
	The sum of *n* terms of an arithmetic sequence	$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right) = \frac{n}{2}(u_1 + u_n)$
	The *n*th term of a geometric sequence	$u_n = u_1 r^{n-1}$
	The sum of *n* terms of a finite geometric sequence	$S_n = \frac{u_1(r^n - 1)}{r-1} = \frac{u_1(1-r^n)}{1-r},\ r \neq 1$
	The sum of an infinite geometric sequence	$S_\infty = \frac{u_1}{1-r},\ \lvert r \rvert < 1$
1.2	Exponents and logarithms	$a^x = b \Leftrightarrow x = \log_a b$, where $a > 0, b > 0, a \neq 1$ $a^x = e^{x \ln a}$ $\log_a a^x = x = a^{\log_a x}$ $\log_b a = \frac{\log_c a}{\log_c b}$
1.3	Combinations	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$
	Permutations	${}^nP_r = \frac{n!}{(n-r)!}$
	Binomial theorem	$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$
1.5	Complex numbers	$z = a + \mathrm{i}b = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{i\theta} = r\,\mathrm{cis}\,\theta$
1.7	De Moivre's theorem	$\left[r(\cos\theta + \mathrm{i}\sin\theta)\right]^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta) = r^n\mathrm{e}^{\mathrm{i}n\theta} = r^n\,\mathrm{cis}\,n\theta$

Topic 2: Functions and equations

2.5	Axis of symmetry of the graph of a quadratic function	$f(x)=ax^2+bx+c \Rightarrow$ axis of symmetry $x=-\frac{b}{2a}$
2.6	Discriminant	$\Delta=b^2-4ac$

Topic 3: Circular functions and trigonometry

3.1	Length of an arc	$l=\theta r$, where θ is the angle measured in radians, r is the radius
	Area of a sector	$A=\frac{1}{2}\theta r^2$, where θ is the angle measured in radians, r is the radius
3.2	Identities	$\tan\theta=\frac{\sin\theta}{\cos\theta}$ $\sec\theta=\frac{1}{\cos\theta}$ $\operatorname{cosec}\theta=\frac{1}{\sin\theta}$
	Pythagorean identities	$\cos^2\theta+\sin^2\theta=1$ $1+\tan^2\theta=\sec^2\theta$ $1+\cot^2\theta=\csc^2\theta$
3.3	Compound angle identities	$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$ $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$ $\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$
	Double angle identities	$\sin 2\theta=2\sin\theta\cos\theta$ $\cos 2\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta$ $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$

3.7	Cosine rule	$c^2 = a^2 + b^2 - 2ab\cos C$; $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$
	Sine rule	$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$
	Area of a triangle	$A = \dfrac{1}{2}ab\sin C$

Topic 4: Vectors

4.1	Magnitude of a vector	$\lvert \boldsymbol{v} \rvert = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
	Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
	Coordinates of the midpoint of a line segment with endpoints (x_1, y_1, z_1), (x_2, y_2, z_2)	$\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2} \right)$
4.2	Scalar product	$\boldsymbol{v} \cdot \boldsymbol{w} = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \cos\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
	Angle between two vectors	$\cos\theta = \dfrac{v_1 w_1 + v_2 w_2 + v_3 w_3}{\lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert}$
4.3	Vector equation of a line	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$
	Parametric form of the equation of a line	$x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$
	Cartesian equations of a line	$\dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n}$

4.5	Vector product	$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$ where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ $\lvert \boldsymbol{v} \times \boldsymbol{w} \rvert = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \sin\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$
	Area of a triangle	$A = \frac{1}{2}\lvert \boldsymbol{v} \times \boldsymbol{w} \rvert$ where $\boldsymbol{v}$ and $\boldsymbol{w}$ form two sides of a triangle
4.6	Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$
	Equation of a plane (using the normal vector)	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
	Cartesian equation of a plane	$ax + by + cz = d$

Topic 5: Statistics and probability

5.1	**Population parameters**	Let $n = \sum_{i=1}^{k} f_i$
	Mean μ	$\mu = \frac{\sum_{i=1}^{k} f_i x_i}{n}$
	Variance σ^2	$\sigma^2 = \frac{\sum_{i=1}^{k} f_i (x_i - \mu)^2}{n} = \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2$
	Standard deviation σ	$\sigma = \sqrt{\frac{\sum_{i=1}^{k} f_i (x_i - \mu)^2}{n}}$
5.2	Probability of an event A	$\mathrm{P}(A) = \frac{n(A)}{n(U)}$
	Complementary events	$\mathrm{P}(A) + \mathrm{P}(A') = 1$
5.3	Combined events	$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$
	Mutually exclusive events	$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$

5.4	Conditional probability	$\mathrm{P}(A\mid B)=\dfrac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$
	Independent events	$\mathrm{P}(A\cap B)=\mathrm{P}(A)\,\mathrm{P}(B)$
	Bayes' theorem	$\mathrm{P}(B\mid A)=\dfrac{\mathrm{P}(B)\,\mathrm{P}(A\mid B)}{\mathrm{P}(B)\,\mathrm{P}(A\mid B)+\mathrm{P}(B')\,\mathrm{P}(A\mid B')}$ $\mathrm{P}(B_i\mid A)=\dfrac{\mathrm{P}(B_i)\mathrm{P}(A\mid B_i)}{\mathrm{P}(B_1)\mathrm{P}(A\mid B_1)+\mathrm{P}(B_2)\mathrm{P}(A\mid B_2)+\mathrm{P}(B_3)\mathrm{P}(A\mid B_3)}$
5.5	Expected value of a discrete random variable X	$\mathrm{E}(X)=\mu=\sum x\,\mathrm{P}(X=x)$
	Expected value of a continuous random variable X	$\mathrm{E}(X)=\mu=\int_{-\infty}^{\infty} x\,f(x)\,\mathrm{d}x$
	Variance	$\mathrm{Var}(X)=\mathrm{E}(X-\mu)^2=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$
	Variance of a discrete random variable X	$\mathrm{Var}(X)=\sum(x-\mu)^2\,\mathrm{P}(X=x)=\sum x^2\,\mathrm{P}(X=x)-\mu^2$
	Variance of a continuous random variable X	$\mathrm{Var}(X)=\int_{-\infty}^{\infty}(x-\mu)^2\,f(x)\,\mathrm{d}x=\int_{-\infty}^{\infty}x^2\,f(x)\,\mathrm{d}x-\mu^2$
5.6	Binomial distribution	$X\sim\mathrm{B}(n,p)\ \Rightarrow\ \mathrm{P}(X=x)=\binom{n}{x}p^x(1-p)^{n-x},\ x=0,1,\ldots,n$
	Mean	$\mathrm{E}(X)=np$
	Variance	$\mathrm{Var}(X)=np(1-p)$
	Poisson distribution	$X\sim\mathrm{Po}(m)\ \Rightarrow\ \mathrm{P}(X=x)=\dfrac{m^x\mathrm{e}^{-m}}{x!},\ x=0,1,2,\ldots$
	Mean	$\mathrm{E}(X)=m$
	Variance	$\mathrm{Var}(X)=m$
5.7	Standardized normal variable	$z=\dfrac{x-\mu}{\sigma}$

Topic 6: Calculus

6.1	Derivative of $f(x)$	$y = f(x) \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = \lim_{h \to 0}\left(\dfrac{f(x+h) - f(x)}{h}\right)$
6.2	Derivative of x^n	$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$
	Derivative of $\sin x$	$f(x) = \sin x \Rightarrow f'(x) = \cos x$
	Derivative of $\cos x$	$f(x) = \cos x \Rightarrow f'(x) = -\sin x$
	Derivative of $\tan x$	$f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$
	Derivative of e^x	$f(x) = \mathrm{e}^x \Rightarrow f'(x) = \mathrm{e}^x$
	Derivative of $\ln x$	$f(x) = \ln x \Rightarrow f'(x) = \dfrac{1}{x}$
	Derivative of $\sec x$	$f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$
	Derivative of $\csc x$	$f(x) = \csc x \Rightarrow f'(x) = -\csc x \cot x$
	Derivative of $\cot x$	$f(x) = \cot x \Rightarrow f'(x) = -\csc^2 x$
	Derivative of a^x	$f(x) = a^x \Rightarrow f'(x) = a^x(\ln a)$
	Derivative of $\log_a x$	$f(x) = \log_a x \Rightarrow f'(x) = \dfrac{1}{x \ln a}$
	Derivative of $\arcsin x$	$f(x) = \arcsin x \Rightarrow f'(x) = \dfrac{1}{\sqrt{1-x^2}}$
	Derivative of $\arccos x$	$f(x) = \arccos x \Rightarrow f'(x) = -\dfrac{1}{\sqrt{1-x^2}}$
	Derivative of $\arctan x$	$f(x) = \arctan x \Rightarrow f'(x) = \dfrac{1}{1+x^2}$
	Chain rule	$y = g(u)$, where $u = f(x) \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u} \times \dfrac{\mathrm{d}u}{\mathrm{d}x}$
	Product rule	$y = uv \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = u\dfrac{\mathrm{d}v}{\mathrm{d}x} + v\dfrac{\mathrm{d}u}{\mathrm{d}x}$
	Quotient rule	$y = \dfrac{u}{v} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

6.4	Standard integrals	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$ $\int \frac{1}{x} \, dx = \ln\lvert x \rvert + C$ $\int \sin x \, dx = -\cos x + C$ $\int \cos x \, dx = \sin x + C$ $\int e^x \, dx = e^x + C$ $\int a^x \, dx = \frac{1}{\ln a} a^x + C$ $\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$ $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin\left(\frac{x}{a}\right) + C, \quad \lvert x \rvert < a$
6.5	Area under a curve Volume of revolution (rotation)	$A = \int_a^b y \, dx$ or $A = \int_a^b x \, dy$ $V = \int_a^b \pi y^2 \, dx$ or $V = \int_a^b \pi x^2 \, dy$
6.7	Integration by parts	$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ or $\int u \, dv = uv - \int v \, du$

Options

Topic 7: Statistics and probability

Further mathematics HL topic 3

7.1 (3.1)	Probability generating function for a discrete random variable X	$G(t) = \mathrm{E}(t^x) = \sum_x \mathrm{P}(X = x)t^x$ $\mathrm{E}(X) = G'(1)$ $\mathrm{Var}(X) = G''(1) + G'(1) - (G'(1))^2$
7.2 (3.2)	Linear combinations of two independent random variables X_1, X_2	$\mathrm{E}(a_1X_1 \pm a_2X_2) = a_1\mathrm{E}(X_1) \pm a_2\mathrm{E}(X_2)$ $\mathrm{Var}(a_1X_1 \pm a_2X_2) = a_1^2\,\mathrm{Var}(X_1) + a_2^2\,\mathrm{Var}(X_2)$
7.3 (3.3)	Sample statistics	
	Mean $\overline{x}$	$\overline{x} = \frac{\sum_{i=1}^{k} f_i x_i}{n}$
	Variance s_n^2	$s_n^2 = \frac{\sum_{i=1}^{k} f_i (x_i - \overline{x})^2}{n} = \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - \overline{x}^2$
	Standard deviation s_n	$s_n = \sqrt{\frac{\sum_{i=1}^{k} f_i (x_i - \overline{x})^2}{n}}$
	Unbiased estimate of population variance s_{n-1}^2	$s_{n-1}^2 = \frac{n}{n-1} s_n^2 = \frac{\sum_{i=1}^{k} f_i (x_i - \overline{x})^2}{n-1} = \frac{\sum_{i=1}^{k} f_i x_i^2}{n-1} - \frac{n}{n-1}\overline{x}^2$
7.5 (3.5)	Confidence intervals	
	Mean, with known variance	$\overline{x} \pm z \times \frac{\sigma}{\sqrt{n}}$
	Mean, with unknown variance	$\overline{x} \pm t \times \frac{s_{n-1}}{\sqrt{n}}$
7.6 (3.6)	Test statistics	
	Mean, with known variance	$z = \frac{\overline{x} - \mu}{\sigma / \sqrt{n}}$

	Mean, with unknown variance	$t = \dfrac{\overline{x} - \mu}{s_{n-1} / \sqrt{n}}$
7.7 (3.7)	Sample product moment correlation coefficient	$r = \dfrac{\sum_{i=1}^{n} x_i y_i - n\overline{x}\,\overline{y}}{\sqrt{\left(\sum_{i-1}^{n} x_i^2 - n\overline{x}^2\right)\left(\sum_{i=1}^{n} y_i^2 - n\overline{y}^2\right)}}$
	Test statistic for $\mathrm{H_0}$: $\rho = 0$	$t = r\sqrt{\dfrac{n-2}{1-r^2}}$
	Equation of regression line of *x* on *y*	$x - \overline{x} = \left(\dfrac{\sum_{i=1}^{n} x_i y_i - n\overline{x}\,\overline{y}}{\sum_{i=1}^{n} y_i^2 - n\overline{y}^2}\right)(y - \overline{y})$
	Equation of regression line of *y* on *x*	$y - \overline{y} = \left(\dfrac{\sum_{i=1}^{n} x_i y_i - n\overline{x}\,\overline{y}}{\sum_{i=1}^{n} x_i^2 - n\overline{x}^2}\right)(x - \overline{x})$

Topic 8: Sets, relations and groups

Further mathematics HL topic 4

8.1 (4.1)	De Morgan's laws	$(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$

Topic 9: Calculus

Further mathematics HL topic 5

9.5 (5.5)	Euler's method	$y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$, where *h* is a constant (step length)
	Integrating factor for $y' + P(x)y = Q(x)$	$\mathrm{e}^{\int P(x)\mathrm{d}x}$

9.6 (5.6)	Maclaurin series	$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}\,f''(0) + \ldots$
	Taylor series	$f(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2!}\,f''(a) + \ldots$
	Taylor approximations (with error term $R_n(x)$)	$f(x) = f(a) + (x-a)\,f'(a) + \ldots + \frac{(x-a)^n}{n!}\,f^{(n)}(a) + R_n(x)$
	Lagrange form	$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, where c lies between a and x
	Maclaurin series for special functions	$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \ldots$ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

Topic 10: Discrete mathematics

Further mathematics HL topic 6

10.7 (6.7)	Euler's formula for connected planar graphs	$v - e + f = 2$, where v is the number of vertices, e is the number of edges, f is the number of faces
	Planar, simple, connected graphs	$e \le 3v - 6$ for $v \ge 3$ $e \le 2v - 4$ if the graph has no triangles

Formulae for distributions

Topics 5.6, 5.7, 7.1, further mathematics HL topic 3.1

Discrete distributions

Distribution	Notation	Probability mass function	Mean	Variance
Geometric	$X \sim \text{Geo}(p)$	pq^{x-1} for $x = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{q}{p^2}$
Negative binomial	$X \sim \text{NB}(r,\ p)$	$\binom{x-1}{r-1} p^r q^{x-r}$ for $x = r,\ r+1, \ldots$	$\frac{r}{p}$	$\frac{rq}{p^2}$

Continuous distributions

Distribution	Notation	Probability density function	Mean	Variance
Normal	$X \sim \text{N}(\mu,\ \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} \text{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	μ	σ^2

Topic 1: Linear algebra

1.2	Determinant of a 2×2 matrix	$\boldsymbol{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det \boldsymbol{A}=\lvert \boldsymbol{A} \rvert = ad-bc$
	Inverse of a 2×2 matrix	$\boldsymbol{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \boldsymbol{A}^{-1}=\frac{1}{\det \boldsymbol{A}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},\ ad \neq bc$
	Determinant of a 3×3 matrix	$\boldsymbol{A}=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \Rightarrow \det \boldsymbol{A}=a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$