CHAPTER 1

Roots of quadratic equations

Learning objectives

After studying this chapter, you should:

- know the relationships between the sum and product of the roots of a quadratic equation and the coefficients of the equation
- be able to manipulate expressions involving $\alpha + \beta$ and $\alpha\beta$
- be able to form equations with roots related to a given quadratic equation.

In this chapter you will be looking at quadratic equations with particular emphasis on the properties of their solutions or roots.

1.1 The relationships between the roots and coefficients of a quadratic equation

As you have already seen in the C1 module, any quadratic equation will have **two roots** (even though one may be a repeated root or the roots may not even be real). In this section you will be considering some further properties of these two roots.

Suppose you know that the two solutions of a quadratic equation are $x = 2$ and $x = -5$ and you want to find a quadratic equation having 2 and -5 as its roots.

The method consists of working backwards, i.e. following the steps for solving a quadratic equation but in reverse order.

Now if $x = 2$ and $x = -5$ are the solutions then the equation could have been factorised as

$$(x - 2)(x + 5) = 0.$$

Expanding the brackets gives

$$x^2 + 3x - 10 = 0.$$

This is a quadratic equation with roots 2 and -5.

Actually, any multiple of this equation will also have the same roots, e.g.

$2x^3 + 6x - 20 = 0$

$3x^2 + 9x - 30 = 0$

$\frac{1}{2}x^2 + \frac{3}{2}x - 5 = 0$

The general case

Consider the most general quadratic equation $ax^2 + bx + c = 0$ and suppose that the two solutions are $x = \alpha$ and $x = \beta$.

Now if α and β are the roots of the equation then you can 'work backwards' to generate the original equation.

A quadratic with the two solutions $x = \alpha$ and $x = \beta$ is

$$(x - \alpha)(x - \beta) = 0.$$

Any multiple of this equation such as $kx^2 - k(\alpha + \beta)x + k\alpha\beta = 0$ will also have roots α and β.

Expanding the brackets gives

$$x^2 - \alpha x - \beta x + \alpha\beta = 0$$
$$\Rightarrow \quad x^2 - (\alpha + \beta)x + \alpha\beta = 0 \quad [1]$$

The most general quadratic equation is $ax^2 + bx + c = 0$ and this can easily be written in the same form as equation [1].

You can divide $ax^2 + bx + c = 0$ throughout by a giving

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad [2]$$

The coefficients of x^2 in [1] and in [2] are now both equal to 1.

Recall that the coefficient of x^2 is the number in front of x^2.

Since [1] and [2] have the same roots, α and β, and are in the same form, you can write

$$x^2 - (\alpha + \beta)x + \alpha\beta \equiv x^2 + \frac{b}{a}x + \frac{c}{a}$$

The $\equiv$ sign means 'identically equal to'.

Equating coefficients of x gives

$$-(\alpha + \beta) = \frac{b}{a} \quad \Rightarrow \quad \alpha + \beta = -\frac{b}{a}$$

The coefficients of x and the constant terms must be equal.

Equating constant terms gives

$$\alpha\beta = \frac{c}{a}$$

When the quadratic equation $ax^2 + bx + c = 0$ has roots α and β:

- The sum of the roots, $\alpha + \beta = -\frac{b}{a}$;
- and the product of roots, $\alpha\beta = \frac{c}{a}$.

Notice it is fairly easy to express

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ as

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

Worked example 1.1

Write down the sum and the product of the roots for each of the following equations:

(a) $2x^2 + 12x - 3 = 0$, **(b)** $x^2 - 8x + 5 = 0$.

Solution

(a) $2x^2 + 12x - 3 = 0$

Here $a = 2$, $b = 12$ and $c = -3$.

Take careful note of the signs.

The sum of roots, $\alpha + \beta = -\frac{b}{a} = -\frac{12}{2} = -6$.

The product of roots, $\alpha\beta = \frac{c}{a} = \frac{-3}{2} = -1\frac{1}{2}$.

(b) $x^2 - 8x + 5 = 0$

Here $a = 1$, $b = -8$ and $c = 5$.

The sum of roots, $\alpha + \beta = -\frac{b}{a} = -\frac{-8}{1} = 8$.

Notice the double negative.

The product of roots, $\alpha\beta = \frac{c}{a} = \frac{5}{1} = 5$.

Worked example 1.2

Find the sum and the product of the roots of each of the following quadratic equations:

(a) $4x^2 + 8x = 5$, **(b)** $x(x - 4) = 6 - 2x$.

Solution

Neither equation is in the form $ax^2 + bx + c = 0$ and so the first thing to do is to get them into this standard form.

(a) $4x^2 + 8x = 5$

$\Rightarrow 4x^2 + 8x - 5 = 0$

Now in the form $ax^2 + bx + c = 0$.

In this case $a = 4$, $b = 8$ and $c = -5$.

The sum of roots, $\alpha + \beta = -\frac{b}{a} = -\frac{8}{4} = -2$.

The product of roots, $\alpha\beta = \frac{c}{a} = \frac{-5}{4} = -\frac{5}{4}$.

(b) $x(x - 4) = 6 - 2x$

Expand the brackets and take everything onto the LHS.

$\Rightarrow x^2 - 4x + 2x - 6 = 0$

$\Rightarrow x^2 - 2x - 6 = 0$

Now in the standard form.

Here $a = 1$, $b = -2$ and $c = -6$.

The sum of roots, $\alpha + \beta = -\frac{b}{a} = -\frac{-2}{1} = 2$.

The product of roots, $\alpha\beta = \frac{c}{a} = \frac{-6}{1} = -6$.

Worked example 1.3

Write down equations with integer coefficients for which:

(a) sum of roots = 4, product of roots = −7,

(b) sum of roots = −4, product of roots = 15,

(c) sum of roots = $-\frac{3}{5}$, product of roots = $-\frac{1}{2}$.

Solution

(a) A quadratic equation can be written as

$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$x^2 - (4)x + (-7) = 0$

$\Rightarrow \quad x^2 - 4x - 7 = 0$

(b) Using

$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

gives $x^2 - (-4)x + (15) = 0$

$\Rightarrow \quad x^2 + 4x + 15 = 0$

Again great care must be taken with the signs.

(c) Using

$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

gives $x^2 - \left(-\frac{3}{5}\right)x + \left(-\frac{1}{2}\right) = 0$

$\Rightarrow \quad x^2 + \frac{3}{5}x - \frac{1}{2} = 0$

Some of the coefficients are fractions not integers. You can eliminate the fractions by multiplying throughout by 10 or (2 × 5).

$\Rightarrow \quad 10x^2 + 10\left(\frac{3}{5}\right)x - 10\left(\frac{1}{2}\right) = 0$

$\Rightarrow \quad 10x^2 + 6x - 5 = 0$

You now have integer coefficients.

EXERCISE 1A

1 Find the sum and product of the roots for each of the following quadratic equations:

(a) $x^2 + 4x - 9 = 0$ **(b)** $2x^2 - 3x - 5 = 0$

(c) $2x^2 + 10x - 3 = 0$ **(d)** $1 + 2x - 3x^2 = 0$

(e) $7x^2 + 12x = 6$ **(f)** $x(x - 2) = x + 6$

(g) $x(3 - x) = 5x - 2$ **(h)** $ax^2 - a^2x - 2a^3 = 0$

(i) $ax^2 + 8a = (1 - 2a)x$ **(j)** $\dfrac{4}{x + 5} = \dfrac{x - 3}{2}$

2 Write down a quadratic equation with:

(a) sum of roots = 5, product of roots = 8,

(b) sum of roots = −3, product of roots = 5,

(c) sum of roots = 4, product of roots = −7,

(d) sum of roots = −9, product of roots = −4,

(e) sum of roots = $\frac{1}{4}$, product of roots = $\frac{2}{5}$,

(f) sum of roots = $-\frac{2}{3}$, product of roots = 4,

(g) sum of roots = $\frac{3}{5}$, product of roots = 0,

(h) sum of roots = k, product of roots = $3k^2$,

(i) sum of roots = $k + 2$, product of roots = $6 - k^2$,

(j) sum of roots = $-(2 - a^2)$, product of roots = $(a + 7)^2$.

1

1.2 Manipulating expressions involving α and β

As you have seen in the last section, given a quadratic equation with roots α and β you can find the values of $\alpha + \beta$ and $\alpha\beta$ without solving the equation.

As you will see in this section, it is useful to be able to write other expressions involving α and β in terms of $\alpha + \beta$ and $\alpha\beta$.

Worked example 1.4

Given that $\alpha + \beta = 4$ and $\alpha\beta = 7$, find the values of:

(a) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$, **(b)** $\alpha^2\beta^2$.

Solution

(a) You need to write this expression in terms of $\alpha + \beta$ and $\alpha\beta$ in order to use the values given in the question.

Now $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\beta + \alpha}{\alpha\beta} = \dfrac{\alpha + \beta}{\alpha\beta}$

$\Rightarrow \quad \dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{4}{7}$

(b) $\alpha^2\beta^2 = (\alpha\beta)^2$

$\Rightarrow \quad \alpha^2\beta^2 = 7^2 = 49$

Two relations which will prove very useful are

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$
$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

Notice how the expressions on the RHS contain combinations of just $\alpha + \beta$ and $\alpha\beta$.

These two results can be proved fairly easily:

$(\alpha + \beta)^2 = (\alpha + \beta)(\alpha + \beta) \Rightarrow (\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

$\Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

as required

and

$(\alpha + \beta)^3 = (\alpha + \beta)(\alpha + \beta)(\alpha + \beta)$

$\Rightarrow \quad (\alpha + \beta)^3 = (\alpha + \beta)(\alpha^2 + 2\alpha\beta + \beta^2)$

$\Rightarrow \quad (\alpha + \beta)^3 = \alpha^3 + 2\alpha^2\beta + \alpha\beta^2 + \alpha^2\beta + 2\alpha\beta^2 + \beta^3$

$\Rightarrow \quad (\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$

$\Rightarrow \quad (\alpha + \beta)^3 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta)$

Take $3\alpha\beta$ as a factor.

$\Rightarrow \quad \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$ as required

Worked example 1.5

Given that $\alpha + \beta = 5$ and $\alpha\beta = -2$, find the values of:

(a) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$, **(b)** $\alpha^3 + \beta^3$, **(c)** $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$.

Solution

(a) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2 + \beta^2}{\alpha\beta} = \dfrac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$

Now it is in terms of $\alpha + \beta$ and $\alpha\beta$.

Substitute the known values of $\alpha + \beta$ and $\alpha\beta$

$$\Rightarrow \quad \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{5^2 - 2(-2)}{-2} = \frac{25 + 4}{-2} = -\frac{29}{2}$$

(b) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$$\Rightarrow \quad \alpha 3 + \beta^3 = 5^3 - 3(-2)(5) = 125 + 30 = 155$$

(c) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} = \dfrac{\beta^2 + \alpha^2}{\alpha^2\beta^2} = \dfrac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$

$$\Rightarrow \quad \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{5^2 - 2(-2)}{(-2)^2} = \frac{25 + 4}{4} = \frac{29}{4}$$

Worked example 1.6

Given that $\alpha + \beta = 5$ and $\alpha\beta = \frac{2}{3}$, find the value of $(\alpha - \beta)^2$.

Solution

$$(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2 = \alpha^2 + \beta^2 - 2\alpha\beta$$

$$= (\alpha + \beta)^2 - 2\alpha\beta - 2\alpha\beta = (\alpha + \beta)^2 - 4\alpha\beta$$

$$\Rightarrow \quad (\alpha - \beta)^2 = 5^2 - 4\left(\tfrac{2}{3}\right) = 25 - \tfrac{8}{3} = 22\tfrac{1}{3}$$

Worked example 1.7

Write the expression $\dfrac{\alpha}{\beta^2} + \dfrac{\beta}{\alpha^2}$ in terms of $\alpha + \beta$ and $\alpha\beta$.

Solution

$$\frac{\alpha}{\beta^2} + \frac{\beta}{\alpha^2} = \frac{\alpha^3 + \beta^3}{\alpha^2\beta^2} = \frac{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}{(\alpha\beta)^2}$$

EXERCISE 1B

1 Write each of the following expressions in terms of $\alpha + \beta$ and $\alpha\beta$:

(a) $\frac{2}{\alpha} + \frac{2}{\beta}$ **(b)** $\frac{1}{\alpha^2\beta} + \frac{1}{\beta^2\alpha}$

(c) $\frac{\alpha}{3\beta} + \frac{\beta}{3\alpha}$ **(d)** $\alpha^2\beta + \beta^2\alpha$

(e) $(2\alpha - 1)(2\beta - 1)$ **(f)** $\frac{\alpha + 5}{\beta} + \frac{\beta + 5}{\alpha}$

2 Given that $\alpha + \beta = -3$ and $\alpha\beta = 9$, find the values of:

(a) $\alpha^3\beta + \beta^3\alpha$, **(b)** $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.

3 Given that $\alpha + \beta = 4$ and $\alpha\beta = 10$, find the values of:

(a) $\alpha^2 + \beta^2$, **(b)** $\alpha^3 + \beta^3$.

4 Given that $\alpha + \beta = 7$ and $\alpha\beta = -2$, find the values of:

(a) $\frac{1}{\beta^2} + \frac{1}{\alpha^2}$, **(b)** $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$.

5 The roots of the quadratic equation $x^2 - 5x + 3 = 0$ are α and β.

(a) Write down the values of $\alpha + \beta$ and $\alpha\beta$.

(b) Hence find the values of:

(i) $(\alpha - 3)(\beta - 3)$, **(ii)** $\frac{\alpha}{\beta^2} + \frac{\beta}{\alpha^2}$.

6 The roots of the equation $x^2 - 4x + 3 = 0$ are α and β. Without solving the equation, find the value of:

(a) $\frac{1}{\alpha} + \frac{1}{\beta}$ **(b)** $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

(c) $\alpha^2 + \beta^2$ **(d)** $\alpha^2\beta + \alpha\beta^2$

(e) $(\alpha - \beta)^2$ **(f)** $(\alpha + 1)(\beta + 1)$

7 The roots of the quadratic equation $x^2 + 4x + 1 = 0$ are α and β.

(a) Find the values of: **(i)** $\alpha + \beta$, **(ii)** $\alpha\beta$.

(b) Hence find the value of:

(i) $(\alpha^2 - \beta)(\beta^2 - \alpha)$, **(ii)** $\frac{\alpha + 2}{\beta + 2} + \frac{\beta + 2}{\alpha + 2}$.

1.3 Forming new equations with related roots

It is often possible to find a quadratic equation whose roots are related in some way to the roots of another given quadratic equation.

Worked example 1.8

The roots of the equation $2x^2 - 5x - 6 = 0$ are α and β.

Find a quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

Solution

$$2x^2 - 5x - 6 = 0$$

$\Rightarrow$ The sum of roots, $\alpha + \beta = -\frac{b}{a} = -\frac{-5}{2} = \frac{5}{2}$.

The product of roots, $\alpha\beta = \frac{c}{a} = \frac{-6}{2} = -3$.

You would be able to write down the 'new' equation with roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ if you could find the sum and the product of the new roots.

The sum of new roots, $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$

$$= \frac{\frac{5}{2}}{-3} = -\frac{5}{6}.$$

The product of new roots, $\frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta}$

$$= \frac{1}{-3} = -\frac{1}{3}.$$

Using $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

the equation with roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is

$$x^2 - \left(-\frac{5}{6}\right)x + \left(-\frac{1}{3}\right) = 0$$

$$\Rightarrow \quad x^2 + \frac{5}{6}x - \frac{1}{3} = 0$$

$$\Rightarrow \quad 6x^2 + 5x - 2 = 0.$$

Multiply through by 6 in order to obtain integer coefficients (because it is often required in an examination question).

The last example illustrates the basic method for forming new equations with roots that are related to the roots of a given equation:

1. Write down the sum of the roots, $\alpha + \beta$, and the product of the roots, $\alpha\beta$, of the given equation.
2. Find the sum and product of the new roots in terms of $\alpha + \beta$ and $\alpha\beta$.
3. Write down the new equation using
   $x^2 - (\text{sum of new roots})x + (\text{product of new roots}) = 0$.

Worked example 1.9

The roots of the equation $x^2 + 7x - 2 = 0$ are α and β.

Find the values of $\alpha^2 + \beta^2$ and $\alpha^2\beta^2$.

Hence, find a quadratic equation whose roots are α^2 and β^2.

Solution

From the equation $x^2 + 7x - 2 = 0$ you have

The sum of roots, $\alpha + \beta = -\dfrac{b}{a} = -\dfrac{7}{1} = -7$.

The product of roots, $\alpha\beta = \dfrac{c}{a} = \dfrac{-2}{1} = -2$.

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (-7)^2 - 2(-2) = 49 + 4 = 53$

and $\alpha^2\beta^2 = (\alpha\beta)^2 = (-2)^2 = 4$.

Now since the roots of the new equation are α^2 and β^2, $\alpha^2 + \beta^2$ and $\alpha^2\beta^2$ are the sum and product of the new roots.

The required equation is

$x^2 - (\text{sum of new roots})x + (\text{product of new roots}) = 0$

$\Rightarrow \quad x^2 - 53x + 4 = 0.$

You could use any variable you choose. For instance, you could write $y^2 - 53y + 4 = 0$.

Worked example 1.10

The roots of the quadratic equation $x^2 + 5x - 3 = 0$ are α and β.

Find a quadratic equation whose roots are α^3 and β^3.

Solution

From the equation $x^2 + 5x - 3 = 0$ you have

The sum of roots, $\alpha + \beta = -\dfrac{b}{a} = -\dfrac{5}{1} = -5$.

The product of roots, $\alpha\beta = \dfrac{c}{a} = \dfrac{-3}{1} = -3$.

The new equation has roots α^3 and β^3.

The sum of new roots, $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= (-5)^3 - 3(-3)(-5)$

$= -125 - 45 = -170.$

The product of new roots, $\alpha^3\beta^3 = (\alpha\beta)^3 = (-3)^3 = -27$.

Using $x^2 - (\text{sum of new roots})x + (\text{product of new roots}) = 0$ the required equation is

$x^2 + 170x - 27 = 0.$

EXERCISE 1C

1 The roots of the equation $x^2 + 6x - 4 = 0$ are α and β. Find a quadratic equation whose roots are α^2 and β^2.

2 The roots of the equation $x^2 + 3x - 5 = 0$ are α and β. Find a quadratic equation whose roots are $\dfrac{2}{\alpha}$ and $\dfrac{2}{\beta}$.

3 The roots of the equation $x^2 - 9x + 5 = 0$ are α and β. Find a quadratic equation whose roots are $\alpha + 2$ and $\beta + 2$.

4 Given that the roots of the equation $2x^2 + 5x - 3 = 0$ are α and β, find an equation with integer coefficients whose roots are $\alpha\beta^2$ and $\alpha^2\beta$.

5 Given that the roots of the equation $3x^2 - 6x + 1 = 0$ are α and β, find a quadratic equation with integer coefficients whose roots are α^3 and β^3.

6 The roots of the equation $2x^2 + 4x - 1 = 0$ are α and β. Find a quadratic equation with integer coefficients whose roots are $\dfrac{\beta}{\alpha}$ and $\dfrac{\alpha}{\beta}$.

7 The roots of the equation $3x^2 - 6x - 2 = 0$ are α and β.

(a) Find the value of $\alpha^2 + \beta^2$.

(b) Find a quadratic equation whose roots are $\alpha^2 + 1$ and $\beta^2 + 1$.

8 The roots of the equation $2x^2 + 7x + 3 = 0$ are α and β. Without solving this equation,

(a) find the value of $\alpha^3 + \beta^3$.

(b) Hence, find a quadratic equation with integer coefficients which has roots $\dfrac{\alpha}{\beta^2}$ and $\dfrac{\beta}{\alpha^2}$.

9 Given that α and β are the roots of the equation $5x^2 - 2x + 4 = 0$, find a quadratic equation with integer coefficients which has roots $\dfrac{1}{\alpha^2}$ and $\dfrac{1}{\beta^2}$.

10 The roots of the equation $x^2 + 4x - 6 = 0$ are α and β. Find an equation whose roots are $\alpha^2 + \beta$ and $\beta^2 + \alpha$.

1.4 Further examples

Sometimes the questions may require you to apply similar techniques to the previous section but without directing you to find the sum and product of the roots.

Worked example 1.11

The roots of the equation $x^2 - kx + 28 = 0$ are α and $\alpha + 3$.
Find the two possible values of k.

Solution

From $x^2 - kx + 28 = 0$ you have

The sum of roots, $\alpha + (\alpha + 3) = -\dfrac{-k}{1} \quad \Rightarrow \quad 2\alpha + 3 = k \qquad [1]$

The product of roots, $\alpha(\alpha + 3) = \dfrac{28}{1} \quad \Rightarrow \quad \alpha^2 + 3\alpha = 28 \quad [2]$

Equations [1] and [2] form a pair of simultaneous equations.

From [1] you have $\alpha = \dfrac{k-3}{2}$ and substituting this into [2] gives

$$\left(\frac{k-3}{2}\right)^2 + 3\left(\frac{k-3}{2}\right) = 28$$

$$\Rightarrow \quad \frac{k^2 - 6k + 9}{4} + \frac{3k - 9}{2} = 28$$

$$\Rightarrow \quad k^2 - 6k + 9 + 6k - 18 = 112$$

$$\Rightarrow \quad k^2 = 121$$

$\Rightarrow$ the two possible values for k are 11 and -11.

1.5 New equations by means of a substitution

Worked example 1.8 is reproduced below.

The roots of the equation $2x^2 - 5x - 6 = 0$ are α and β.

Find a quadratic equation whose roots are $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$.

Since the new equation has roots that are the reciprocals of the original equation, the new equation can be found very quickly by making the substitution $y = \dfrac{1}{x} \quad \Rightarrow \quad x = \dfrac{1}{y}$, which transforms

$2x^2 - 5x - 6 = 0$ into $\dfrac{2}{y^2} - \dfrac{5}{y} - 6 = 0$.

Multiplying throughout by y^2 gives

$2 - 5y - 6y^2 = 0 \quad$ or

$6y^2 + 5y - 2 = 0$.

Check the working in Worked example 1.8 to see which is quicker.

Worked example 1.12

The roots of the equation $x^2 - 3x - 5 = 0$ are α and β.
Find quadratic equations with roots

(a) $\alpha - 3$ and $\beta - 3$, **(b)** α^2 and β^2.

You could solve this question using the sum and product of roots technique of the previous section if you prefer.

Solution

(a) Use the substitution $y = x - 3$ in $x^2 - 3x - 5 = 0$, since the roots in the new equation are 3 less than those in the original equation.

$y = x - 3 \quad \Rightarrow \quad x = y + 3$, so new equation is

$(y + 3)^2 - 3(y + 3) - 5 = 0$

or $\quad y^2 + 6y + 9 - 3y - 9 - 5 = 0$

The new equation is $y^2 + 3y - 5 = 0$.

If you prefer, you can write the new equation as $x^2 + 3x - 5 = 0$ or in terms of any other variable.

(b) This time you need to use the substitution $y = x^2$ in $x^2 - 3x - 5 = 0$ to eliminate x.

Hence, $y - 3x - 5 = 0 \quad$ or $\quad y - 5 = 3x$.

Squaring both sides gives $(y - 5)^2 = 9x^2 = 9y$.

Hence, $y^2 - 10y + 25 = 9y \quad$ or $\quad y^2 - 19y + 25 = 0$.

Alternative solution

Using $\alpha + \beta = 3$ and $\alpha\beta = -5$

(a) Sum of new roots is

$$(\alpha - 3) + (\beta - 3) = \alpha + \beta - 6 = -3$$

Product of new roots is

$$(\alpha - 3)(\beta - 3) = \alpha\beta - 3\alpha - 3\beta + 9$$
$$= \alpha\beta - 3(\alpha + \beta) + 9 = -5$$

Hence new equation is $y^2 + 3y - 5 = 0$.

(b) Sum of new roots is

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$
$$= 9 + 10 = 19$$

Product of new roots is

$$\alpha^2\beta^2 = (\alpha\beta)^2 = 25$$

Hence new equation is $y^2 - 19y + 25 = 0$.

You should see that the answers are the same whichever method you use. If the question directs you to use a particular substitution, e.g. 'Use the substitution $y = x^2$ to find a new equation whose roots are α^2 and β^2', you must use the first method. However, if the question is of the form 'Use the substitution $y = x^2$, or otherwise, to find a new equation whose roots are α^2 and β^2', or simply 'Find an equation whose roots are α^2 and β^2' you may use either technique.

Worked examination question 1.13

The roots of the quadratic equation $x^2 - 3x - 7 = 0$ are α and β.

(a) Write down the values of **(i)** $\alpha + \beta$, **(ii)** $\alpha\beta$.

(b) Find a quadratic equation with integer coefficients whose roots are $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$.

Solution

(a) From the equation $x^2 - 3x - 7 = 0$ you have

The sum of roots, $\alpha + \beta = -\dfrac{b}{a} = -\dfrac{-3}{1} = 3$.

The product of roots, $\alpha\beta = \dfrac{c}{a} = \dfrac{-7}{1} = -7$.

(b) The new equation has roots $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$.

$$\text{The sum of new roots, } \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$
$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$
$$= \frac{3^2 - 2(-7)}{-7} = \frac{9 + 14}{-7} = -\frac{23}{7}$$

The product of new roots, $\dfrac{\alpha}{\beta} \times \dfrac{\beta}{\alpha} = \dfrac{\alpha\beta}{\alpha\beta} = 1$.

Using $x^2 -$ (sum of new roots)$x +$ (product of new roots) $= 0$ the required equation is

$$x^2 - \left(-\frac{23}{7}\right)x + 1 = 0$$
$$\Rightarrow \quad x^2 + \frac{23}{7}x + 1 = 0$$
$$\Rightarrow \quad 7x^2 + 23x + 7 = 0$$

A common mistake is to forget to multiply throughout by 7 to obtain integer coefficients.

EXERCISE 1D

1 The roots of the equation $x^2 + 9x + k = 0$ are α and $\alpha + 1$. Find the value of k.

2 The roots of the quadratic equation $4x^2 + kx - 5 = 0$ are α and $\alpha - 3$. Find the two possible values of the constant k.

3 The roots of the quadratic equation $x^2 + 3x - 7 = 0$ are α and β. Find an equation with integer coefficients which has roots $\dfrac{3}{\alpha}$ and $\dfrac{3}{\beta}$.

4 The roots of the quadratic equation $2x^2 + 5x - 1 = 0$ are α and β. Find an equation with integer coefficients which has roots $\alpha - 2$ and $\beta - 2$.

5 The roots of the quadratic equation $x^2 - 2x - 5 = 0$ are α and β. Find a quadratic equation which has roots $\alpha^2 + 1$ and $\beta^2 + 1$.

6 The roots of the quadratic equation $x^2 + 5x - 7 = 0$ are α and β.

(a) Without solving the equation, find the values of

(i) $\alpha^2 + \beta^2$, **(ii)** $\alpha^3 + \beta^3$.

(b) Determine an equation with integer coefficients which has roots $\dfrac{\alpha^2}{\beta}$ and $\dfrac{\beta^2}{\alpha}$. [A]

7 The roots of the quadratic equation $3x^2 + 4x - 1 = 0$ are α and β.

(a) Without solving the equation, find the values of

(i) $\alpha^2 + \beta^2$, **(ii)** $\alpha^3\beta + \beta^3\alpha$.

(b) Determine a quadratic equation with integer coefficients which has roots $\alpha^3\beta$ and $\beta^3\alpha$. [A]

8 The roots of the quadratic equation $x^2 + 2x + 3 = 0$ are α and β.

(a) Without solving the equation:

(i) write down the value of $\alpha + \beta$ and the value of $\alpha\beta$,

(ii) show that $\alpha^3 + \beta^3 = 10$,

(iii) find the value of $\dfrac{1}{\alpha^3} + \dfrac{1}{\beta^3}$.

(b) Determine a quadratic equation with integer coefficients which has roots $\dfrac{1}{\alpha^3}$ and $\dfrac{1}{\beta^3}$. [A]

9 The roots of the quadratic equation $x^2 - 3x + 1 = 0$ are α and β.

(a) Without solving the equation:

(i) show that $\alpha^2 + \beta^2 = 7$,

(ii) find the value of $\alpha^3 + \beta^3$.

(b) **(i)** Show that $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2$.

(ii) Hence, find the value of $\alpha^4 + \beta^4$.

(c) Determine a quadratic equation with integer coefficients which has roots $(\alpha^3 - \beta)$ and $(\beta^3 - \alpha)$. [A]

10 The roots of the quadratic equation $x^2 + 3x - 2 = 0$ are α and β.

(a) Write down the values of $\alpha + \beta$ and $\alpha\beta$.

(b) Without solving the equation, find the value of:

(i) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$, **(ii)** $\left(\alpha - \dfrac{3}{\beta^2}\right)\left(\beta - \dfrac{3}{\alpha^2}\right)$.

(c) Determine a quadratic equation with integer coefficients which has roots $\alpha - \dfrac{3}{\beta^2}$ and $\beta - \dfrac{3}{\alpha^2}$. [A]

Key point summary

1 When the quadratic equation $ax^2 + bx + c = 0$ has roots α and β: *p2*

- The sum of the roots, $\alpha + \beta = -\dfrac{b}{a}$;
- and the product of roots, $\alpha\beta = \dfrac{c}{a}$.

2 A quadratic equation can be expressed as $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$. *p2*

3 Two useful results are: *p5*

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$.

4 The basic method for forming new equations with roots that are related to the roots of a given equation is: *p8*

1 Write down the sum of the roots, $\alpha + \beta$, and the product of the roots, $\alpha\beta$, of the given equation.
2 Find the sum and product of the new roots in terms of $\alpha + \beta$ and $\alpha\beta$.
3 Write down the new equation using

$$x^2 - (\text{sum of new roots})x + (\text{product of new roots}) = 0$$

Test yourself — What to review

1 Find the sums and products of the roots of the following equations: *Section 1.1*

(a) $x^2 - 6x + 4 = 0$, **(b)** $2x^2 + x - 5 = 0$.

2 Write down the equations, with integer coefficients, where: *Section 1.3*

(a) sum of roots $= -4$, product of roots $= 7$,

(b) sum of roots $= \frac{5}{4}$, product of roots $= -\frac{1}{2}$.

3 The roots of the equation $x^2 + 5x - 6 = 0$ are α and β. Find the values of: *Section 1.2*

(a) $\alpha^2 + \beta^2$ **(b)** $\dfrac{\beta}{\alpha} + \dfrac{\alpha}{\beta}$.

4 Given that the roots of $3x^2 - 9x + 1 = 0$ are α and β, find a quadratic equation whose roots are $\dfrac{1}{\alpha\beta^2}$ and $\dfrac{1}{\alpha^2\beta}$. *Section 1.3*

Test yourself ANSWERS

1 **(a)** sum $= 6$, product $= 4$; **(b)** sum $= -\frac{1}{2}$, product $= -\frac{5}{2}$.

2 **(a)** $x^2 + 4x + 7 = 0$; **(b)** $4x^2 - 5x - 2 = 0$.

3 **(a)** 37; **(b)** $-\frac{37}{6}$.

4 $x^2 - 27x + 27 = 0$.